Determination of Allowable Tolerances for Asymmetries of a Free Rolling Missile

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This paper presents a method for determining tolerances on free rolling missile asymmetries. Practical flight limitations such as a specified maximum lateral aerodynamic load are taken into account. The proposed method can deal with missile dynamics described by differential equations of motion. It is demonstrated for the case of a vehicle having a lateral c.g. and aerodynamic asymmetries. The vehicle tolerance contour with the mass asymmetry as an independent variable is calculated by an optimal parameter search combining the gradient projection technique and the continuation method. For this search, the maximum value of the lateral load and its partial derivatives with respect to the asymmetry parameters must be known. They are obtained by integrating numerically the differential equations of motion and the related sensitivity equations. The method is applied to a numerical example.

Nomenclature

A = maximum allowed value for the lateral force
a_B = aerodynamic asymmetry tolerance, deg
a_g = nondimensional c.g. tolerance
$\bar{c}_g$ = nondimensional c.g. offset, $Y_{c.g.}/d$
C_{l_o} = rolling moment coefficient due to pure roll torque
C_{l_p} = roll damping derivative coefficient, $\partial C_l/\partial(pd/2V)$, rad^{-1}
C_m, C_n = pitching and yawing moment coefficients
C_{m_o}, C_{n_o} = aerodynamic moment coefficients due to asymmetry
C_{m_q} = pitch damping derivative, $\partial C_m/\partial(qd/2V)$, rad^{-1}
C_{m_α} = pitching moment derivative, $\partial C_m/\partial\alpha$, rad^{-1}
$C_{m_{\dot\alpha}}$ = damping derivative coefficient, $\partial C_m/\partial(\dot\alpha d/2V)$, rad^{-1}
C_N = normal force coefficient
C_{N_α} = normal force derivative, $\partial C_N/\partial\alpha m$, rad^{-1}
C_x = axial coefficient
d = reference diameter, m
F = lateral force
F_M = maximum F during the flight
i = $(-1)^{1/2}$
M_y, M_z = pitch and yaw moments, Newton × m
I, I_y, I_z = lateral moments of inertia, $I = I_y = I_z$, kg × m^2
I_x = axial moment of inertia, kg × m^2
J_{xy}, J_{yz}, J_{xz} = products of inertia, kg × m^2
m = mass of vehicle, kg
p, q, r = angular rates about the X, Y, Z axes, rad/sec
Q = dynamic pressure, Newton/m^2
S = reference area, m^2
V = total velocity, m/x
t = time, sec
T = duration flight, sec
x, y, z = inertial reference axes
X, Y, Z = body reference axes (Fig. 1)
X_g, Y_g, Z_g = geometric axes with X_g the axis of geometry symmetry (Fig. 1)
$Y_{c.g.}$ = c.g. offset along Y axis, m
α, β = angles of attack and sideslip
α_A = $-C_{m_o}/C_{m_\alpha}$, rad or deg
β_A = $-C_{n_o}/C_{m_\alpha}$, rad or deg
δ = complex angle of attack, rad
δ_A = $(C_{m_o}{}^2+C_{n_o}{}^2)^{1/2}/C_{m_\alpha}$, rad or deg
Ω = $q+ir$, rad/sec
ρ = atmospheric density, kg/m^3

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Introduction

NUMEROUS works[1-7] have shown that geometrical asymmetries combined with inertial asymmetries may cause the occurrence of large angles of attack during re-entry flights leading to dispersion or structural failure. The determination of tolerance limits for the asymmetries is therefore necessary in design studies. Much analytical work has been devoted to the study of the motion of an asymmetric missile with variable roll rate. Many analytical studies were concerned with the development of approximate analytic solutions for describing the asymmetry induced roll rate behavior. As a result of these analyses, criteria were proposed for estimation of the minimum asymmetries needed for roll resonance lock-in.[3,5-7] Approximate analytic results were also derived for predicting the magnitude of the impact dispersion due to asymmetries.[8-10] The effect of a roll rate reversal on the dispersion was examined in Refs. 8 and 9. A closed-form solution for the dispersion was obtained assuming that the roll rate history can be approximated with a series of linear segments.[10] The digital six-degree-of-freedom computation of missile motion provides another means for investigating the effects of asymmetries. Tolerance limits related to a certain criterion may be deduced from empirical parameter studies based on digital flight simulations.

This paper presents a method for determining asymmetry tolerances subject to practical flight limitations. Similar methods have not appeared in the open literature. To demonstrate the method the following requirement is considered here: the lateral aerodynamic load exerted on the vehicle has to be smaller than a given maximum level during the flight. Tolerances are defined by the minimum values of the asymmetries for which the constraint is violated. A method of computation for the tolerances is developed for the case of a vehicle having a lateral c.g. offset and aerodynamic asymmetries creating pitching and yawing moments at zero angle of attack. The vehicle tolerance contour with the mass asymmetry as an independent variable is calculated by an optimal parameter search. The method is applied to a numerical example.

Analysis

Equations of Motion

The linearized equations of motion for a basically symmetrical vehicle having slight mass, aerodynamic and inertia asymmetries were derived by A. E. Hodapp Jr. and E. L. Clark[4] for a body axis system whose origin is at the c.g. of the body (Fig. 1).

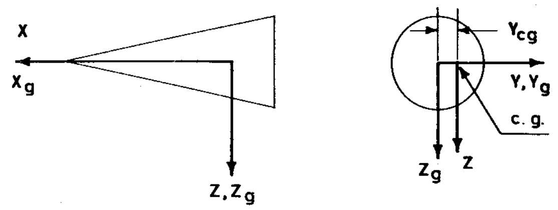

Fig. 1 Coordinate system.

The assumptions used in Ref. 4 were 1) small amplitudes of oscillation, 2) linear aerodynamics, 3) the moment-of-inertia tensor is symmetric with $I = I_x = I_z$, $J_{yz} = 0$, 4) the effects of gravity and of aerodynamic forces due to asymmetry are negligible, and 5) mean straight line trajectory.

We further assume: 1) the c.g. is located on the body-fixed axis Y, 2) negligible Magnus forces and moments, and 3) $J_{xy} = J_{xz} = 0$. With these latter assumptions the equations presented in Ref. 4 reduce to the following set of equations:

Transversal Force Equation

$$\dot{\delta} = -[ip + (QS/mV)(C_x + C_{N_\alpha})]\delta + i\Omega; \quad \delta = \beta + i\alpha \tag{1}$$

Roll Moment Equation

$$\dot{p} = (QSd/I_x)[C_{l_o} + C_{N_\alpha}\bar{c}_g\alpha + C_{l_p}(pd/2V)] \tag{2}$$

Transversal Moment Equation

$$\dot{\Omega} = -i\left(1 - \frac{I_x}{I}\right)p\Omega + \left(\frac{M_y + iM_z}{I}\right); \quad \Omega = q + ir \tag{3}$$

where

$$M_y = QSd\left[C_{m_\alpha}\alpha + C_{m_q}q\frac{d}{2V} + C_{m_{\dot{\alpha}}}\dot{\alpha}\frac{d}{2V} - C_{m_\alpha}\alpha_A\right] \tag{4}$$

$$M_z = QSd\left[-C_{m_\alpha}\beta + C_{m_q}r\frac{d}{2V} - C_{m_{\dot{\alpha}}}\dot{\beta}\frac{d}{2V} + C_{m_\alpha}\beta_A + C_x\bar{c}_g\right] \tag{5}$$

$$\alpha_A = -C_{m_o}/C_{m_\alpha}; \qquad \beta_A = -C_{n_o}/C_{m_\alpha} \tag{5a}$$

$$\bar{c}_g = Y_{\text{c.g.}}/d$$

The sign of C_{n_o} in Eq. (5a) differs from that used in Ref. 4, and the coefficient C_x is used instead of $C_A (C_x = -C_A)$.

The initial conditions of the system of Eqs. (1–5) are

$$\delta(t=0) = \delta_o$$
$$\Omega(t=0) = \Omega_o$$
$$p(t=0) = p_o$$

The time dependencies of the speed V and the dynamic pressure Q are assumed to be known from preliminary mean trajectory calculations.

Definition of Allowable Tolerances on Asymmetries

Let $F_M = \max_t F$ be the maximum lateral aerodynamic load exerted on the vehicle during the flight. Clearly F_M is a function of the asymmetries $\bar{c}_g$, α_A, β_A. A typical requirement on a re-entry flight is

$$F_M < A \tag{6}$$

where A is a given positive quantity. The problem to solve is to derive allowable tolerances on asymmetries from the constraint (6). Tolerances should be defined in a way where the assumed basic axisymmetry of the vehicle is taken into consideration. Therefore tolerances will not be imposed on each one of the parameters α_A, β_A, but on the symmetric quantity

$$\delta_A = \alpha_A{}^2 + \beta_A{}^2$$

The significance of this latter quantity is quite clear. Indeed δ_A is the amplitude of the nonrolling complex trimming angle of attack.

The positive numbers, a_g, a_B are allowable tolerances for $\bar{c}_g$, δ_A, respectively, if two conditions are fulfilled: a) for at least one combination of asymmetries $(\bar{c}_g, \alpha_A)$ such that

$$\bar{c}_g = a_g$$
$$\delta_A = a_B$$

then $F_M = A$; b) for any other combination of asymmetries such that

$$\bar{c}_g < a_g$$
$$\delta_a < a_B$$

the requirement $F_M < A$ is satisfied.

As a consequence of this definition, tolerances satisfy a simple property. Let a_g, a_B be a set of allowable tolerances. Then for fixed a_g and a_j, a_B is the smallest value of δ_A such that condition (a) of the precedent definition is satisfied. Indeed, smaller values than a_B satisfy conditions (b) and, therefore, cannot satisfy condition (a). Similarly, the same result is obtained for a_g. We shall postulate that the reciprocal of this property is also true. That is to say, if, for fixed a_g there exists a lower limit a_B of values of δ_A such that condition (a) is satisfied, then (a_g, a_B) is a set of allowable tolerances. In relation with this hypothesis, arises the following problem: determine α_A, β_A such that

$$\delta_A{}^2 = \alpha_A{}^2 + \beta_A{}^2 \text{ be minimum and}$$
$$F_M(\bar{c}_g = a_g, \alpha_A, \beta_A) = A$$

where a_g is a positive number.

As already mentioned the solution of this problem which will be named T-problem gives a set of allowable tolerances (a_g, a_B) where a_B is a function of (a_g). The information which is of interest for the design of a vehicle is the knowledge of the tolerance contour. This is the graphical representation of the function $a_B(a_g)$ (see Fig. 2). Therefore, it is required to obtain the solution of the T-problem for a physically significant range of values of a_g.

Numerical Analysis

Principles of the method

The T-problem stated in the previous section belongs to the class of minimum problems with an equality constraint. An important feature of this problem in our case is that the function F_M appearing in the constraint is not known explicitly but must be evaluated numerically. The principles of the numerical method for obtaining tolerance contours will now be presented.

The tolerance curve $a_B(a_g)$ will be determined by a set of discrete points $P_i(a_g{}^i, a_B{}^i)$; $i = (1, \ldots, k)$. The determination of the first point $P_o(a_g{}^o, a_B{}^o = 0)$ is described in a later section. The last value of a_g is taken to be zero. The ordinate $a_B{}^i$ of each point P_i is obtained by solving the T-problem for the corresponding given value $a_g{}^i$ of a_g. The numerical method which has been used for solving each T-problem is an iterative method, the gradient projection technique. An excellent description of this

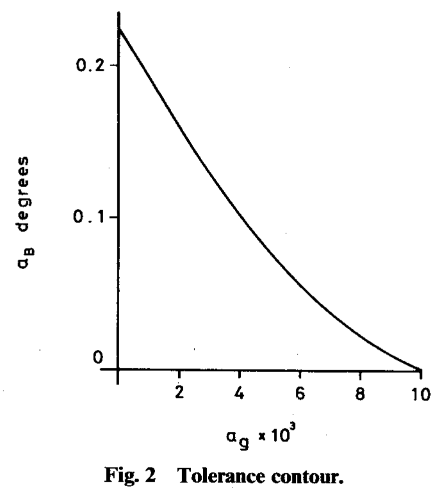

Fig. 2 Tolerance contour.

technique is given by Kelley.[11] As for most iterative methods, the convergence of the numerical process towards the true solution depends crucially on a good first approximation of the unknown parameters. An interesting feature of the current solution is that this requirement is easily satisfied: indeed the solution $(\alpha_A{}^i, \beta_A{}^i)$ associated with the point P_i, is an excellent approximation for determining the next point P_{i+1}, if the interval $|a_g{}^{i+1} - a_g{}^i|$ is small enough. The idea of determining the tolerance contour in this fashion, each computed point being a starting solution for the next desired point, belongs to the continuation method in its discrete numerical version.[12] Generally it is necessary to introduce an auxiliary parameter in the equations of the problem for using this method. However, this operation is not necessary here because this parameter, a_g, already figures in the definition of the problem. This facilitates the application of the continuation method to the current problem.

The continuation method has been applied to the resolution of nonlinear equations,[13] boundary-value problems, and recently to the parameter identification problem.[14] It seems that it was not used until now for solving a minimum problem with constraint. The reason the continuation method is not employed in its continuous version is that computation of second-order partial derivatives of the function F_M would have been necessary and the complexity of the calculations would have been increased.

Determination of a Current Solution

The solution of the T-problem by the gradient projection technique will now be described. A value $a_g{}^i$ of a_g is chosen. The solution of the T-problem for $a_g = a_g{}^{i-1}$ is assumed to be known and gives a first approximation of the unknown parameters α_A, β_A.

In order to start the gradient technique procedure a solution of the constraint equation

$$F_M(\bar{c}_g = a_g{}^i, \alpha_A, \beta_A) = A \tag{7}$$

must be determined. Clearly calculation of this solution might be done in many different ways because there are two unknowns α_A, β_A and only one equation. It has been found efficient to operate in the following manner. As a starting approximation, the point M with coordinates

$$\bar{c}_g = a_g{}^i$$
$$\alpha_A = \alpha_A{}^{i-1}$$
$$\beta_A = \beta_A{}^{i-1}$$

is chosen, where $(\alpha_A{}^{i-1}, \beta_A{}^{i-1})$ are obtained from the solution of the T-problem for $a_g = a_g{}^{i-1}$. The solution of the constraint equation (7) will be searched along the line (L) passing through the point M and directed along the gradient of the function F_M at the point M. Geometrically the line (L) is the perpendicular drawn from point M to the surface defined by Eq. (7). The coordinates of the current point of line (L) are

$$\alpha_A = \alpha_A{}^{i-1} + \lambda \frac{\partial F_M}{\partial \alpha_A}(a_g{}^i, \alpha_A{}^{i-1}, \beta_A{}^{i-1}) \tag{8}$$

$$\beta_A = \beta_A{}^{i-1} + \lambda \frac{\partial F_M}{\partial \beta_A}(a_g{}^i, \alpha_A{}^{i-1}, \beta_A{}^{i-1}) \tag{9}$$

where λ is a scalar quantity.

The line of search being defined, the one-dimensional nonlinear equation

$$F_M(\lambda) - A = 0 \tag{10}$$

must be solved. Equation (10) may be solved by applying the classical Newton-Raphson method. This method yields the following recurrence formula for the successive approximations of the solution of Eq. (10)

$$\lambda_{n+1} = \lambda_n \frac{F_M(\lambda_n) - A}{(dF_M/d\lambda)(\lambda_n)}; \quad \lambda_o = 0 \tag{11}$$

where

$$\frac{dF_M}{d\lambda} = \left(\frac{\partial F_M}{\partial \alpha_A}\right)_{\lambda=0} \frac{\partial F}{\partial \alpha_A} + \left(\frac{\partial F_M}{\partial \beta_A}\right)_{\lambda=0} \left(\frac{\partial F}{\partial \beta_A}\right) \tag{12}$$

The iteration process is stopped when the difference between two successive values of λ becomes smaller than a prescribed value.

As already noticed, since neither the function F_M nor its parameter derivatives are known analytically, they must be computed numerically for each set of parameters. The method of computation of function F_M and its parameter derivatives is developed in a subsequent section of the paper.

The solution $(\alpha_A{}^{i,N}, \beta_A{}^{i,N})$ of Eq. (10) obtained by the Newton-Raphson method will now be modified in order to satisfy the minimum condition of the T-problem. This change will be done by applying the gradient projection technique. Geometrically, the new point (α_A, β_A) is located along the projection of the gradient of the function $(\alpha_A{}^2 + \beta_A{}^2)$ on the plane tangential to the surface $F_M = A$ at the point $(\alpha_A{}^{i,N}, \beta_A{}^{i,N})$. Applying results given by Kelley,[11] the coordinates of the current point of the projection of the gradient are obtained:

$$\alpha_A = \alpha_A{}^{i,N} - a_m \Delta\sigma \tag{13}$$

$$\beta_A = \beta_A{}^{i,N} - a_n \Delta\sigma \tag{14}$$

where $\Delta\sigma$ is a scalar quantity and

$$a_m = 2\alpha_A{}^{i,N} + \mu\, \partial F_M/\partial \alpha_A \tag{15}$$

$$a_n = 2\beta_A{}^{i,N} + \mu\, \partial F_M/\partial \beta_A \tag{16}$$

$$\mu = -\frac{2\alpha_A{}^{i,N}(\partial F_M/\partial \alpha_A) + 2\beta_A{}^{i,N}(\partial F_M/\partial \beta_A)}{(\partial F_M/\partial \alpha_A)^2 + (\partial F_M/\partial \beta_A)^2} \tag{17}$$

The partial derivatives $\partial F_M/\partial \alpha_A$, $\partial F_M/\partial \beta_A$ are evaluated at the point

$$(a_g{}^i, \alpha_A{}^{i,N}, \beta_A{}^{i,N})$$

The step size in Eqs. (13) and (14) can be optimized by writing that

$$(\alpha_A{}^{i,N+1})^2 + (\beta_A{}^{i,N+1})^2$$

is minimum. Expressing this condition enables the calculation of the optimum value of $\Delta\sigma$

$$\Delta\sigma_{\text{opt}} = \frac{\alpha_A{}^{i,N} a_m + \beta_A{}^{i,N} a_n}{(a_m)^2 + (a_n)^2} \tag{18}$$

The new values of (α_A, β_A) are then obtained by substituting in Eqs. (13) and (14) this optimum value of $\Delta\sigma$. Now the Newton-Raphson process is repeated in order to restore the constraint if the constraint has been violated beyond a prescribed tolerance. The gradient procedure follows, and this cycle is continued until any prescribed degree of convergence is reached. A new value $a_g{}^{i+1}$ of a_g is then assumed and the whole process is repeated. The computation is stopped when the final value of a_g is obtained.

Determination of the Initial Solution

Clearly an initial solution is needed for performing the sequence of calculations previously described. This initial solution will be obtained by setting $a_B = 0$. With this value of a_B the problem reduces to the determination of the minimum value of $|\bar{c}_g|$ such that the constraint is satisfied. A first approximation of this value of $|\bar{c}_g|$ may be determined graphically by computing the function F_M for a series of values of $\bar{c}_g$. The solution of the one-dimensional equation

$$F_M(\bar{c}_g, \alpha_A = 0, \beta_A = 0) - A = 0 \tag{19}$$

which is closest to the graphical solution can be obtained by any standard numerical technique for finding zeros of nonlinear equations. For instance a bisecant technique was used.

It may be observed that the solution

$$a_B = 0, \quad a_g = a_g{}^o$$

gives the absolute maximum allowable tolerance for $\bar{c}_g$.

Remark

In the numerical procedure which has been described, the first approximation of the solution of the T-problem for $a_g = a_g{}^i$ was chosen equal to the solution of the T-problem at the previous value of a_g, a_g^{i-1}. An eventually better approximation may be obtained by an extrapolation based on a small number of precedent solutions. For instance, a two-point extrapolation yields the values

$$x^i = x^{i-1} + \frac{x^{i-1} - x^{i-2}}{a_g^{i-1} - a_g^{i-2}}(a_g^i - a_g^{i-1})$$

where x is any of the searched parameters (α_A, β_A).

The procedure which employs such an extrapolation scheme remains essentially the same as before and it will not be detailed here. The introduction of this extrapolation scheme may hopefully reduce the number of iterations for solving each T-problem.

Computation of the Function F_M and its Partial Derivatives

In this part of the paper, the method of computation of the implicit function F_M of the parameters $\bar{c}_g$, α_A, β_A and of its partial derivatives with respect to these parameters is described. Numerical integration of the equations of motion by a standard Runge-Kutta technique gives the time history of the aerodynamic force F. The maximum value F_M of F over the flight is obtained by comparison of the values of F at successive integration times. Let t_m denote the time at which F is maximum and c any of the parameters (c_g, α_A, β_A) on which F_M may depend. We shall show a property of the partial derivative of F_M with respect to c.

One has, by definition

$$F_M(c) = \max_{0 \leq t \leq T} F(t, c) = F(t_m, c)|_{t=t_m} \tag{20}$$

It is assumed that F is a continuous and differentiable function of time. If t_m is different from zero and T

$$\frac{\partial F}{\partial t}(t, c) = 0|_{t=t_m} \tag{21}$$

Equation (21) defines t_m as an implicit function of c when c varies so that

$$F_M(c) = F[t_m(c), c] \tag{22}$$

Differentiating Eq. (22) with respect to c

$$\frac{\partial F_M}{\partial c} = \left(\frac{\partial F}{\partial t}\right)_{t=t_m} \frac{\partial t_m}{\partial c} + \frac{\partial F}{\partial c}(t, c)|_{t=t_m} \tag{23}$$

Using Eq. (21), Eq. (23) reduces to

$$\partial F_M / \partial c = (\partial F / \partial c)(t, c)|_{t=t_m} \tag{24}$$

This result is obviously true also when t_m equals zero or T since $\partial t_m / \partial c$ becomes null in Eq. (23).

The partial derivatives of the function F_M will be computed by applying Eq. (24). Now the partial derivative $\partial F/\partial c(t, c)$ must be evaluated. This derivative may be expressed as

$$\frac{\partial F}{\partial c} = \frac{1}{2}\rho V^2 \frac{\partial |\delta|}{\partial c} \tag{25}$$

differentiating with respect to c

$$\delta = |\delta| e^{i\gamma} \tag{26}$$

with respect to c, yields

$$\frac{\partial \delta}{\partial c} = \left(\frac{\partial |\delta|}{\partial c} + i\frac{\partial \gamma}{\partial c}\right) e^{i\gamma} \tag{27}$$

or

$$\frac{|\delta|}{\delta}\frac{\partial \delta}{\partial c} = \frac{\partial |\delta|}{\partial c} + i\frac{\partial \gamma}{\partial c} \tag{28}$$

which yields

$$\frac{\partial |\delta|}{\partial c} = \text{Real}\left(\frac{|\delta|}{\delta}\frac{\partial \delta}{\partial c}\right) \tag{29}$$

Thus the partial derivatives of the complex angle-of-attack δ must be evaluated. Here a technique well known in the control field is applied which was used recently for the determination of aerodynamic derivatives from free-flight data.[15] δ appears in the set of the differential equations (1–5) as a dependent variable. By partial differentiating Eqs. (1–5) with respect to c, differential equations of the c—derivatives of δ and the other dependent variables are obtained. These equations are called sensitivity equations. Numerical integration of the sensitivity equations together with the motion equations yields the required parameter derivatives of δ. Substitution of the computed value of $\partial\delta/\partial c$ at $t = t_m$ in Eq. (25) yields the required partial derivatives of the function F_M. To illustrate the partial differentiation of a differential equation, take $c = \alpha_A$ and differentiate the complex angle-of-attack equation (1) with respect to α_A. Denoting

$$\frac{\partial \delta}{\partial \alpha_A} = \delta_m; \qquad \frac{\partial \Omega}{\partial \alpha_A} = \Omega_m; \qquad \frac{\partial p}{\partial \alpha_A} = p_m$$

It will be assumed that the orders of differentiation can be reversed

$$\dot{\delta}_m = \frac{\partial}{\partial \alpha_A}(\dot{\delta}) \tag{30}$$

Differentiation of Eq. (1) with respect to α_A yields the equation

$$\dot{\delta}_m = -\left[ip + \frac{(C_x + C_{N_\alpha})QS}{mV}\right]\delta_m - ip_m\delta + i\Omega_m \tag{31}$$

which is a linear differential equation with variable coefficients.

Partial differentiation of Eqs. (1–5) with respect to any of the parameters (α_A, β_A) may be performed in the same way. The sensitivity equations derived for each one of the parameters α_A, δ_A appear in the Appendix.

Numerical Example

The tolerances for the asymmetries $\bar{c}_g$, α_A, β_A of an unfinned vehicle were determined by using the preceding method. The initial conditions and other parameters considered for the computation were

$V_o = 6000$ m/sec	$\gamma_o = -30°$	$h_o = 40{,}000$ m
$P_o = 30$ rad/sec	$\Omega_o = 0$	$\beta_o = 0$, $\alpha_o = 0.3°$
$C_{N_\alpha} = 1.5$/rad	$C_{m_\alpha} = -0.082$/rad	$C_{m_q} = -3.45$/rad
$m = 100$ kg	$I_x = 0.58$ kg × m^2	$I = 5.82$ kg × m^2
$d = 0.5$ m	$S = 0.196$ m^2	$C_{l_o} = 0$, $C_{l_p} = 0$

These vehicle characteristics were obtained by taking typical values of missile data published in the literature.

The maximum lateral load was fixed as $A = 8700$ N. The results of the computations are presented in Figs. 2 and 3. The tolerance contour $a_B(a_g)$ is plotted in Fig. 2. Figure 3 shows the path followed by the optimal solution (α_A, β_A) as a_g varies from its maximum value until zero.

This numerical example demonstrates the property of the present technique of providing a converging iterative solution to the family of T-problems defined previously. This procedure may require a large number of digital flight simulations for computing the tolerance curve $a_B(a_g)$. For example, about 100

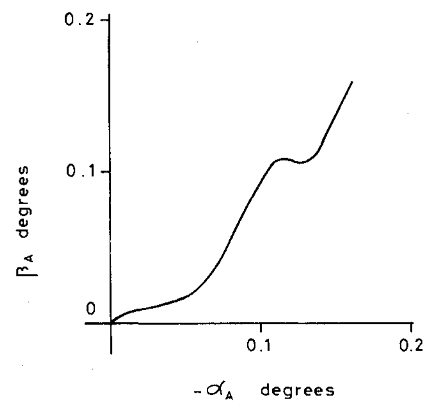

Fig. 3 Optimal trajectory in the plane (α_A, β_A).

iterations were necessary in the solution of the numerical application presented here. It is, however, doubtful that the tolerance curve might be obtained by employing a parametric study using an equal number of flight simulations for establishing the dependence of F_M on the asymmetry magnitudes. The reason for this is that the variation of F_M with the asymmetry parameters α_A, β_A is complicated, as seen for example, in Fig. 8 of Ref. 3.

Conclusions

A new method for determining the tolerance contour of a free rolling vehicle having mass and aerodynamic asymmetries has been described. The contour is computed in a continuous fashion by employing a numerical optimization technique combining the gradient projection technique and the continuation method. This method can be used with the missile dynamics described by the differential equations of motion.

With this method, tolerances are fixed according to specific performance or design requirements. For instance, a maximum allowed value for the lateral aerodynamic load was considered in this paper, however other criteria may be taken into account such as a maximum allowed dispersion. Therefore the present method can be a useful versatile tool for design studies.

The numerical approach presented in this paper can be extended for computing tolerances of vehicles having other mass asymmetries such as products of inertia.

Appendix: Sensitivity Equations

For any function f depending on time and on the asymmetries, the following notation is defined:

$$f_m = \partial f/\partial \alpha_A, \quad f_m = \partial f/\partial \beta_A, \quad \dot{f} = \partial f/\partial t$$

Assuming that f has second-order partial derivatives, the order of differentiation may be reversed, resulting in

$$\dot{f}_m = \partial \dot{f}/\partial \alpha_A, \quad \dot{f}_n = \partial \dot{f}/\partial \beta_A$$

The sensitivity equations are obtained by partial differentiating the equations of motion (1–5) with respect to α_A, β_A. Since the initial conditions associated with Eqs. (1–5) are independent of the asymmetries, the initial conditions associated with the sensitivity equations are all null.

α_A—**Sensitivity Equations**

$$\dot{\delta}_m = -\left[ip + \frac{QS}{mV}(C_x + C_{N_\alpha})\right]\delta_m - ip_m\delta + i\Omega_m; \quad \delta_m = \beta_m + i\alpha_m$$

$$\dot{p}_m = \frac{QSd}{I^x}\left[C_{N_\alpha}\bar{c}_g\alpha_m + C_{l_p}\frac{p_m d}{2V}\right]$$

$$\dot{\Omega}_m = -i\left(1 - \frac{I_x}{I}\right)(p_m\Omega + p\Omega_m) + \frac{M_{y_m} + M_{z_m}}{I}; \quad \Omega_m = q_m + ir_m$$

$$M_{y_m} = QSd\left[C_{m_\alpha}\alpha_m + C_{m_q}q_m\frac{d}{2V} + C_{m_{\dot{\alpha}}}\dot{\alpha}_m\frac{d}{2V} - C_{m_\alpha}\right]$$

$$M_{z_m} = QSd\left[-C_{m_\alpha}\beta_m + C_{m_q}r_m\frac{d}{2V} - C_{m_{\dot{\alpha}}}\dot{\beta}_m\frac{d}{2V}\right]$$

β_A—**Sensitivity Equations**

$$\dot{\delta}_n = -\left[ip + \frac{QS}{mV}(C_x + C_{N_\alpha})\right]\delta_n - ip_n\delta + i\Omega_n; \quad \delta_n = \beta_n + i\alpha_n$$

$$\dot{p}_n = \frac{QSd}{I_x}\left[C_{N_\alpha}c_g\alpha_n + C_{l_p}\frac{p_n d}{2V}\right]$$

$$\dot{\Omega}_n = i\left(1 - \frac{I_x}{I}\right)(p_n\Omega + p\Omega_n) + \frac{M_{y_n} + iM_{z_n}}{I}; \quad \Omega_n = q_n + ir_n$$

$$M_{y_n} = QSd\left[C_{m_\alpha}\alpha_n + C_{m_q}q_n\frac{d}{2V} + C_{m_{\dot{\alpha}}}\frac{d}{2V}\right]$$

$$M_{z_n} = QSd\left[-C_{m_\alpha}\beta_n + C_{m_q}r_n\frac{d}{2V} - C_{m_{\dot{\alpha}}}\dot{\beta}_n\frac{d}{2V} + C_{m_\alpha}\right]$$

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